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CURVES OF CONSTANT BREADTH ACCORDING TO TYPE-2 BISHOP FRAME IN E^3

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Abstract. In this paper, we study the curves of constant breadth according to type-2 Bishop frame in the 3-dimensional Euclidean Space E^3. Moreover some characterizations of these curves are obtained.

1. Introduction

In 1780, L. Euler studied curves of constant breadth in the plane [3]. Thereafter, this issue investigated by many geometers [2, 4, 12]. Constant breadth curves are an important subject for engineering sciences, especially, in cam designs [17]. M. Fujiwara introduced constant breadth for space curves and surfaces [4]. D. J. Struik published some important publications on this subject [16]. O. Kose expressed some characterizations for space curves of constant breadth in Euclidean 3-space[10] and M. Sezer researched space curves of constant breadth and obtained a criterion for these curves [15]. A. Magden and O. Kose obtained constant breadth curves in Euclidean 4-space [11]. Characterizations for spacelike curves of constant breadth in Minkowski 4-space were given by M. Kazaz et al. [9]. S. Yilmaz and M. Turgut studied partially null curves of constant breadth in semi-Riemannian space [18]. The properties of these curves in 3-dimensional Galilean space were given by D. W. Yoon [20]. H. Gun Bozok and H. Oztekin investigated an explicit characterization of mentioned curves according to Bishop frame in 3-dimensional Euclidean space [5]. The curve of constant breadth on the sphere studied by W. Blaschke [2]. Furthermore, the method related to the curves of constant breadth for the kinematics of machinery was given by F. Reuleaux [14].

L. R. Bishop defined Bishop frame, which is known alternative or parallel frame of the curves with the help of parallel vector fields [1]. Then, S. Yilmaz and M. Turgut examined a new version of the Bishop frame which is called type-2 Bishop frame [19]. Thereafter, E. Ozyilmaz studied classical differential geometry of curves according to type-2 Bishop trihedra [13].

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In this paper, we used the theory of the curves with respect to type-2 Bishop frame. Then, we gave some characterizations for curves of constant breadth according to type-2 Bishop frame.

2. Preliminaries

The standard flat metric of 3-dimensional Euclidean space E^3 is given by

$$\langle\, ,\, \rangle : dx_1^2 + dx_2^2 + dx_3^2 \tag{2.1}$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. For an arbitrary vector x in E^3, the norm of this vector is defined by $\|x\| = \sqrt{\langle x, x\rangle}$. α is called a unit speed curve, if $\langle \alpha', \alpha' \rangle = 1$. Suppose that $\{t, n, b\}$ is the moving Frenet–Serret frame along the curve α in E^3. For the curve α, the Frenet-Serret formulae can be given as

$$\begin{aligned} t' &= \kappa n \\ n' &= -\kappa t + \tau b \\ b' &= -\tau n \end{aligned} \tag{2.2}$$

where

$$\langle t, t\rangle = \langle n, n\rangle = \langle b, b\rangle = 1,$$
$$\langle t, n\rangle = \langle t, b\rangle = \langle n, b\rangle = 0.$$

and here, $\kappa = \kappa(s) = \|t'(s)\|$ and $\tau = \tau(s) = -\langle n, b'\rangle$. Furthermore, the torsion of the curve α can be given

$$\tau = \frac{[\alpha', \alpha'', \alpha''']}{\kappa^2}.$$

Along the paper, we assume that $\kappa \neq 0$ and $\tau \neq 0$.

Bishop frame is an alternative approachment to define a moving frame. Assume that $\alpha(s)$ is a unit speed regular curve in E^3. The type-2 Bishop frame of the $\alpha(s)$ is expressed as [19]

$$\begin{aligned} N_1' &= -k_1 B, \\ N_2' &= -k_2 B, \\ B' &= k_1 N_1 + k_2 N_2. \end{aligned} \tag{2.3}$$

The relation matrix may be expressed as

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} \sin\theta(s) & -\cos\theta(s) & 0 \\ \cos\theta(s) & \sin\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ B \end{bmatrix}. \tag{2.4}$$

where $\theta(s) = \int_0^s \kappa(s)\,ds$. Then, type-2 Bishop curvatures can be defined in the following

$$k_1(s) = -\tau(s)\cos\theta(s),$$
$$k_2(s) = -\tau(s)\sin\theta(s).$$

On the other hand,

$$\theta' = \kappa = \frac{\left(\frac{k_2}{k_1}\right)'}{1+\left(\frac{k_2}{k_1}\right)^2}.$$

The frame $\{N_1, N_2, B\}$ is properly oriented, τ and $\theta(s) = \int_0^s \kappa(s)\,ds$ are polar coordinates for the curve α . Then, $\{N_1, N_2, B\}$ is called type-2 Bishop trihedra and k_1, k_2 are called Bishop curvatures.

The characterizations of inclined curves in E^n is given [7] and [8] as follows

Theorem 1. *α is an inclined curve in $E^n \Leftrightarrow \sum_{i=1}^{n-2} H_i^2 = const$ and α is an inclined curve in $E^{n-1} \Leftrightarrow \det\left(V_1', V_2', ..., V_n'\right) = 0$.*

Theorem 2. *Let $M \subset E^3$ is a curve given by (I, α) chart. Then M is an inclined curve if and only if $H(s) = \frac{k_1(s)}{k_2(s)}$ is constant for all $s \in I$.*

3. Curves of Constant Breadth According to type-2 Bishop Frame in E^3

Let $X = \vec{X}(s)$ be a simple closed curve in E^3. These curves will be denoted by (C). The normal plane at every point P on the curve meets the curve at a single point Q other than P. The point Q is called the opposite point of P. Considering a curve α which have parallel tangents $\vec{T}$ and $\vec{T}^*$ in opposite points X and X^* of the curve as in [4]. A simple closed curve of constant breadth which have parallel tangents in opposite directions can be introduced by

$$X^*(s) = X(s) + m_1(s) N_1 + m_2(s) N_2 + m_3(s) B \tag{3.1}$$

where X and X^* are opposite points and N_1, N_2, B denote the type-2 Bishop frame in E^3 space. If N_1 is taken instead of tangent vector and differentiating equation (3.1) we have

$$\begin{aligned}\frac{dX^*}{ds} = \frac{dX^*}{ds^*}\frac{ds^*}{ds} = N_1^*\frac{ds^*}{ds} &= \left(1 + \frac{dm_1}{ds} + m_3 k_1\right) N_1 \\ &+ \left(\frac{dm_2}{ds} + m_3 k_2\right) N_2 \\ &+ \left(\frac{dm_3}{ds} - m_1 k_1 - m_2 k_2\right) B\end{aligned} \tag{3.2}$$

where k_1 and k_2 are the first and the second curvatures of the curve, respectively [6]. Since $N_1^* = -N_1$, we obtain

$$\begin{aligned} \frac{ds^*}{ds} + \frac{dm_1}{ds} + m_3 k_1 + 1 &= 0, \\ \frac{dm_2}{ds} + m_3 k_2 &= 0, \\ \frac{dm_3}{ds} - m_1 k_1 - m_2 k_2 &= 0. \end{aligned} \tag{3.3}$$

Suppose that ϕ is the angle between the tangent of the curve (C) at point $X(s)$ with a given fixed direction and $\frac{d\phi}{ds} = k_1$, then the equation (3.3) can be written as

$$\begin{aligned} \frac{dm_1}{d\phi} &= -m_3 - f(\phi), \\ \frac{dm_2}{d\phi} &= -\rho k_2 m_3, \\ \frac{dm_3}{d\phi} &= m_1 + \rho k_2 m_2, \end{aligned} \tag{3.4}$$

where $f(\phi) = \rho + \rho^*$, $\rho = \frac{1}{k_1}$ and $\rho^* = \frac{1}{k_1^*}$ denote the radius of curvatures at X and X^*, respectively. If we consider equation (3.4), we get

$$\begin{aligned} \frac{k_1}{k_2} m_1''' &+ \left(\frac{k_1}{k_2}\right)' m_1'' + \left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right) m_1' + \left(\frac{k_1}{k_2}\right)' m_1 \\ &+ \left(\frac{k_1}{k_2}\right) f(\phi)'' + \left(\frac{k_1}{k_2}\right)' f(\phi)' + \left(\frac{k_2}{k_1}\right) f(\phi) = 0 \end{aligned} \tag{3.5}$$

This equation is a characterization for X^*. If the distance between the opposite points of (C) and (C^*) is constant, then

$$\|X^* - X\|^2 = m_1^2 + m_2^2 + m_3^2 = l^2, \; l \in \mathbb{R}.$$

Hence, we write

$$m_1 \frac{dm_1}{d\phi} + m_2 \frac{dm_2}{d\phi} + m_3 \frac{dm_3}{d\phi} = 0 \tag{3.6}$$

By considering system (3.4), we obtain

$$m_1 \left(\frac{dm_1}{d\phi} + m_3 \right) = 0. \tag{3.7}$$

Thus we can write $m_1 = 0$ or $\frac{dm_1}{d\phi} = -m_3$. Then, we consider these situations with some subcases.

Case 1. If $\frac{dm_1}{d\phi} = -m_3$, then $f(\phi) = 0$. So, (C^*) is translated by the constant vector

$$u = m_1 N_1 + m_2 N_2 + m_3 B \tag{3.8}$$

of (C). Here, let us solve the equation (3.5), in some special cases.

Case 1.1 Let X be an inclined curve. Then the equation (3.5) can be written as follows,

$$\frac{d^3 m_1}{d\phi^3} + \left(1 + \frac{k_2^2}{k_1^2}\right) \frac{dm_1}{d\phi} = 0. \tag{3.9}$$

The general solution of this equation is

$$m_1 = c_1 + c_2 \cos \sqrt{1 + \frac{k_2^2}{k_1^2}} \phi + c_3 \sin \sqrt{1 + \frac{k_2^2}{k_1^2}} \phi \tag{3.10}$$

And therefore, we have m_2 and m_3, respectively,

$$m_2 = \frac{k_2}{k_1} \left(c_2 \cos \sqrt{1 + \frac{k_2^2}{k_1^2}} \phi \right) + \frac{k_2}{k_1} \left(c_3 \sin \sqrt{1 + \frac{k_2^2}{k_1^2}} \phi \right) \tag{3.11}$$

$$m_3 = c_2 \sqrt{1 + \frac{k_2^2}{k_1^2}} \sin \sqrt{1 + \frac{k_2^2}{k_1^2}} \phi - c_3 \sqrt{1 + \frac{k_2^2}{k_1^2}} \cos \sqrt{1 + \frac{k_2^2}{k_1^2}} \phi \tag{3.12}$$

where c_1 and c_2 are real numbers.

Corollary 1. Position vector of X^* can be formed by the equations (3.10), (3.11) and (3.12). Also the curvature of X^* is obtained as

$$k_1^* = -k_1. \tag{3.13}$$

Case 2. $m_1 = 0$. Then, considering equation (3.5) we get

$$\left(\frac{k_1}{k_2}\right) f(\phi)'' + \left(\frac{k_1}{k_2}\right)' f(\phi)' + \left(\frac{k_2}{k_1}\right) f(\phi) = 0 \tag{3.14}$$

Case 2.1 Suppose that X is an inclined curve. The equation (3.14) can be rewrite as

$$f(\phi)'' + \left(\frac{k_2}{k_1}\right)^2 f(\phi) = 0. \tag{3.15}$$

So, the solution of above differential equation is

$$f(\phi) = L_1 \cos \frac{k_2}{k_1} \phi + L_2 \sin \frac{k_2}{k_1} \phi \tag{3.16}$$

where L_1 and L_2 are real numbers. Using above equation we obtain

$$m_2 = L_1 \sin \frac{k_2}{k_1}\phi - L_2 \cos \frac{k_2}{k_1}\phi \tag{3.17}$$

$$m_3 = -L_1 \cos \frac{k_2}{k_1}\phi - L_2 \sin \frac{k_2}{k_1}\phi = -\rho - \rho^* \tag{3.18}$$

And therefore the curvature of X^* is obtained as

$$k_1^* = \frac{1}{L_1 \cos \frac{k_2}{k_1}\phi + L_2 \sin \frac{k_2}{k_1}\phi - \frac{1}{k_1}} \tag{3.19}$$

And distance between the opposite points of (C) and (C^*) is

$$\|X - X^*\| = L_1^2 + L_2^2 = const. \tag{3.20}$$

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