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S_4 Solution of the Transport Equation for Eigenvalues: Isotropic, Forward and Backward Scattering in a Slab

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Abstract: The neutron transport equation is solved numerically for monoenergetic neutrons in a finite homogeneous slab with backward and forward scattering for the eigenvalue spectrum. The forward-backward-isotropic (FBI) scattering kernel is chosen for representing the neutron scattering in transport equation. Then, the transport equation is converted into a discrete ordinates form by using the integral transform technique with the even-order Gauss-Legendre quadrature set. Finally, the eigenvalues are calculated for a medium from weakly absorbing to highly scattering condition using various values of the scattering, backward and forward scattering parameters. Gauss-Legendre quadrature sets are used for all calculations and the calculated eigenvalues are given in the tables.

***Key words**:* Transport Equation, Forward and Backward Scattering, Discrete Ordinates, Eigenvalues, Gauss-Legendre Quadrature.

Transport Denkleminin Özdeğerler için S_4Çözümü: Dilimde İzotropik, İleri ve Geri Saçılma

Özet: İleri ve geri saçılmalı sonlu homojen bir dilimde tek enerjili nötronların özdeğer spektrumu için nötron transport denklemi nümerik olarak çözülmüştür. Transport denklemindeki nötron saçılmasını temsilen, ileri-geri-izotropic (FBI) saçılma fonksiyonu tercih edilmiştir. Daha sonra, çift-mertebeli Gauss-Legendre kuadratür seti ile integral dönüşüm tekniği kullanılarak transport denklemi diskret-ordinatlar haline dönüştürülmüştür. Son olarak, faklı saçılma, ileri ve geri saçılma parametreleri kullanılarak zayıf yutulmalı bir ortamdan kuvvetli saçılmalı bir ortama kadar özdeğerler hesaplanmıştır. Bütün hesaplamalarda Gauss-Legendre kuadratür setleri kullanılmış ve hesaplanan özdeğerler çizelgelerde verilmiştir.

***Anahtar kelimeler**:* Transport Denklemi, İleri ve Geri Saçılma, Diskret Ordinatlar, Özdeğerler, Gauss-Legendre Kuadratürü.

1. Introduction

The calculation of the eigenvalues is known to be the first attempt for the numerical solution of the transport equation. As an example, the collision parameter c_0, the number of secondary neutrons per collision, is known as the essential eigenvalue and it designates some concepts of a reactor such as diffusion length, diffusion coefficient and buckling.

The neutron transport equation describes the number of neutrons and their interactions in a system. The scattering function taking a part in transport equation describes the types and the number of interactions of neutrons in the system. There are many scattering functions used for the solution of the transport equation in slab, spherical or cylindrical geometries. Among them, the forward-backward-isotropic (FBI) scattering kernel is one of the most powerful and the most commonly used one for especially isotropic scattering [1,2]. Although it does not contain the anisotropic scattering, it can be said to correspond satisfactory approximation for many cases.

The stochastic methods such as Monte Carlo and source iteration are developed for the solution of the transport equation before deterministic methods (polynomial expansion based techniques, integral transform). The probabilistic approach is used in writing the transport equation in discrete ordinates form (S_N) because of the nature of the macroscopic cross-section [3-5].

In this study, the discrete ordinates form of the neutron transport equation is investigated for the solution of the eigenvalues of monoenergetic neutrons. The calculated eigenvalues are obtained by using the fourth order approximation (S_4) of Legendre quadrature set for various values of the forward and backward scattering parameters together with c_0.

2. Material and Method

The stationary transport equation for monoenergetic neutrons can be written in a closed form as,

$$\Omega\cdot\nabla\psi(r,\Omega)+\sigma_T\psi(r,\Omega)=\int_{\Omega'}\psi(r,\Omega')\,\sigma_S(\Omega'\cdot\Omega)\,\mathrm{d}\Omega'+Q_0(x)/2\,, \tag{1}$$

where $\psi(r,\Omega)$ is the neutron angular flux at position r travelling in direction Ω, $\sigma_S(\Omega'\cdot\Omega)$ is the differential scattering cross-section and Q_0 is the internal source [5]. In many of the deterministic methods developed for the solution of the transport equation, it is customary to expand the differential scattering cross-section in terms of the Legendre polynomials since the definition interval of it is the same with the cosine of the scattering angle. In this study, the scattering function is assumed to be of the form as forward-backward-isotropic (FBI) scattering model,

$$\sigma_S(\Omega'\cdot\Omega)=\frac{(1-\alpha-\beta)\sigma_S}{4\pi}+\frac{\alpha\sigma_S}{2\pi}\delta(\Omega'\cdot\Omega-1)+\frac{\beta\sigma_S}{2\pi}\delta(\Omega'\cdot\Omega+1)\,. \tag{2}$$

Here α and β are the parameters varied over the range of $0 \le \alpha, \beta \le 1$, $\alpha + \beta \le 1$ and denote the forward and backward scattering probabilities in a collision, respectively; δ is the Dirac delta function [1,2]. When this scattering function is inserted into Equation 1 one can obtain,

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \sigma_T \psi(x,\mu) = \frac{\sigma_s}{2}(1-\alpha-\beta)\int_{-1}^{1} \psi(x,\mu')\,\mathrm{d}\mu' + \sigma_S\left[\alpha\psi(x,\mu) + \beta\psi(x,-\mu)\right] + \frac{Q_0(x)}{2} \tag{3}$$

$0 \le x \le a,\ -1 \le \mu \le 1$. This equation can be written in discrete ordinates (S_N) form if the integral is written in an integral transform,

$$\mu_m \frac{\mathrm{d}\psi_m(x)}{\mathrm{d}x} + \sigma_T \psi_m(x) = \frac{\sigma_{S0}(1-\alpha-\beta)}{2}\sum_{n=1}^{N} \psi_n(x)\omega_n + \sigma_S\left[\alpha\psi_m(x) + \beta\psi_{N-m+1}(x)\right] + \frac{Q_0(x)}{2} \tag{4}$$

$m = 1,\ldots,N$. Here ω_n is the Gauss-Legendre quadrature weight or the weighting factor for direction μ_n, i.e. the roots of the Nth order Legendre polynomials.

The general solution for Equation 4 can be written as the sum of the homogeneous and the particular solutions,

$$\psi_m(x) = \psi_m^p(x) + \psi_m^h(x)\,. \tag{5}$$

A spatially constant particular solution can be derived from Equation 4of the form as,

$$\psi_m^p(x) = \frac{Q_0}{2\sigma_T(1-c_0)}, \qquad 0 \le a \le x, \qquad 1 \le m \le N\,, \tag{6}$$

where $c_0 = \sigma_{S0}/\sigma_T$. The method of separation of variables can be used to determine the homogeneous solution $\psi_m^h(x)$ of Equation 4 [5],

$$\psi_m^h(x) = H_m(\alpha,\beta,\nu)\exp(\sigma_T x/\nu), \qquad 0 \le a \le x, \qquad 1 \le m \le N\,. \tag{7}$$

Substituting Equation 7 into the homogeneous part of Equation 4, one can obtain an expression for the angular part of the neutron angular flux;

$$H_m(\nu,\alpha,\beta) = \frac{\nu c_0(1-\alpha-\beta)\left[\nu - \mu_m + \nu c_0(\beta-\alpha)\right]}{2\left\{\left[\nu + \mu_m - \nu\alpha c_0\right]\left[\nu - \mu_m - \nu\alpha c_0\right] - \nu^2\beta^2 c_0^2\right\}}\sum_{n=1}^{N} H_n(\nu,\alpha,\beta)\omega_n\,, \tag{8}$$

When Equation 8 is multiplied by ω_m and then summed over all *m*, an equation for the eigenvalues can be obtained:

$$\frac{1}{2}\sum_{m=1}^{N}\frac{vc_0(1-\alpha-\beta)\left[v-\mu_m+vc_0(\beta-\alpha)\right]}{\left\{\left[v+\mu_m-v\alpha c_0\right]\left[v-\mu_m-v\alpha c_0\right]-v^2\beta^2c_0^2\right\}}\omega_m=1, \qquad v\neq-\mu_m. \tag{9}$$

Equation 9 is the dispersion relation and the roots $v_k, 1\leq k\leq N$, of it are the eigenvalues of the S_N equations and they are lying symmetrically about the origin for any c_0 satisfying $0\leq c_0\leq 1$.

3. Results

S_N transport equation with Gauss-Legendre quadrature set is studied for the solution of the eigenvalue problem for one-speed neutrons in a finite homogeneous slab. First the forward-backward-isotropic scattering kernel given in Equation 2 is used in transport equation which is then written in the form of S_N by using the integral transform technique with the even-order Gauss-Legendre quadrature set. The roots, i.e. the eigenvalues $v_k, 1\leq k\leq N$, are calculated up to the forth order approximation from the dispersion relation Equation 9 [4,5]. It is certainly possible to increase the approximation order in this or other related problems. However, it is found to be necessary to do so in this study as an introduction to the method and the problem. The calculated eigenvalues are tabulated in the tables for isotropic, forward and backward scatterings separately for various values of c_0, α and β.

In Table 1, the eigenvalues are calculated for forward scattering with various values of the c_0 ranging from 0.30 to 2.0. Similarly, they are given for backward scattering and forward-backward scattering in Table 2 and 3, respectively. Lastly, the simplest application, i.e. the results obtained from isotropic scattering is given in Table 4. It can be easily seen from the tables that the eigenvalues obtained for forward scattering are some more than the eigenvalues obtained for backward scattering, as expected. The same behavior is also seen in the case of criticality calculations.

Table 1. Eigenvaluespectrumforforwardscattering (α = 0.3, β = 0.0)

N	c_0 = 0.30	c_0 = 0.60	c_0 = 0.99	c_0 = 1.20	c_0 = 2.00
2	±0.7233853	±1.0080973	±6.8859158	±1.6137431*i**	±0.9128709*i*
4	±0.4225452	±0.5489234	±0.7671046	±0.8927136	±1.5681429
	±1.1402322	±5.0790103*i*	±0.7834813*i*	±0.6280986*i*	±0.4763710*i*

*$i = \sqrt{-1}$

Table 2. Eigenvaluespectrumforbackwardscattering (α = 0.0, β = 0.3)

N	c_0 = 0.30	c_0 = 0.60	c_0 = 0.99	c_0 = 1.20	c_0 = 2.00
2	±0.6609629	±0.8403658	±5.0695497	±1.1070186*i**	±0.4564355*i*
4	±0.3860830	±0.4575912	±0.5647579	±0.6123964	±0.7840715
	±1.0418393	±4.2339431	±0.5768147	±0.4308721*i*	±0.2381855*i*

Table 3. Eigenvalue spectrum for forward and backward scattering (α = 0.3, β = 0.3)

N	c_0 = 0.30	c_0 = 0.60	c_0 = 0.99	c_0 = 1.20	c_0 = 2.00

2	±0.6900656	±0.9128709	±5.7735027	±1.2909944*i***	±0.5773503*i*
4	±0.4052639	±0.5216467	±0.8442230	±1.1216802	±1.5087602*i*
	±1.0475330	±1.7538811	±0.8760174*i*	±0.5981692*i*	±0.3234112*i*

Table 4. Eigenvalue spectrum for isotropic scattering (α = 0.0, β = 0.0)

N	c_0 = 0.30	c_0 = 0.60	c_0 = 0.99	c_0 = 1.20	c_0 = 2.00
2	±0.6900656	±0.9128709	±5.7735027	±1.2909944*i**	±0.5577350*i*
4	±0.3998789	±0.4705368	±0.5398162	±0.5622476	±0.6034654
	±1.1576255	±1.3912910*i*	±0.5478575*i*	±0.4400834*i*	±0.2801003*i*

4. Conclusion and Comment

This study is thought to be the first step for other studies related in transport theory. In addition, the method and thus the problem can be said to be extended to more comprehensive situations in order to clarify the unspecified points in neutron transport theory. The method was successfully applied to the transport equation in slab geometry by Anlı and Öztürk et al. for the calculation of the neutron scalar flux [6,7].

Although the case of anisotropy is not added to this study, the present situation can be said to be satisfactory for many cases in transport theory. However, it is planned to investigate the eigenvalue problem in the case of anisotropic scattering together with the forward and backward scattering in future works.

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