Bull. Korean Math. Soc. **50** (2013), No. 2, pp. 445–449
http://dx.doi.org/10.4134/BKMS.2013.50.2.445

SEMIGROUP PRESENTATIONS FOR CONGRUENCES ON GROUPS

Gonca Ayık and Basri Çalışkan

Abstract. We consider a congruence ρ on a group G as a subsemigroup of the direct product $G \times G$. It is well known that a relation ρ on G is a congruence if and only if there exists a normal subgroup N of G such that $\rho = \{(s,t) : st^{-1} \in N\}$. In this paper we prove that if G is a finitely presented group, and if N is a normal subgroup of G with finite index, then the congruence $\rho = \{(s,t) : st^{-1} \in N\}$ on G is finitely presented.

1. Introduction

Finite presentability of semigroup constructions has been widely studied in recent years (see, for example [1, 2, 6, 8, 9]). One construction is an extension of a semigroup by a congruence. Let S and T be semigroups and let ρ be a congruence on S. If S/ρ is isomorphic to T, then S is called an *extension of T by ρ*. There is a similar construction in group theory. An *extension of a group H by a group N* is a group G having N as a normal subgroup and $G/N \cong H$. It is known that if H and N are both finitely presented groups, then the extension of them is finitely presented (see [7, Corollary 10.2]). Recently, it is proved in [3] that, for given a semigroup S and a congruence ρ on S, if ρ is finitely presented as a subsemigroup of the direct product $S \times S$, then S and S/ρ are finitely presented. In [3] finite presentability of ρ on a finitely presented infinite semigroup is an open problem. More recently, for inverse semigroups S and T, and for a surjective homomorphism $\pi : S \to T$ with kernel K which is a congruence on S, it is showed in [4] that how to the obtain a presentation for K from a given a presentation for S and vice versa. It is also investigated in [4] the relationship between finite presentability of inverse semigroups and their kernels.

Let G be a group and let N be a normal subgroup of G with finite index. Then it is known that if G is finitely presented, then N is also finitely presented

Received September 22, 2011.
2010 *Mathematics Subject Classification.* 20M05.
Key words and phrases. congruence, normal subgroup, semigroup presentation.
This work is supported by Research Foundation of Çukurova University.

(see [7, Corollary 9.1]). However the analog of this result is not true for semigroups. For example, consider the free monogenic semigroup $S = \langle x \mid \rangle$ and $\rho = S \times S$. It is known that $\rho = S \times S$ is not finitely generated as a semigroup although S is finitely presented and S/ρ is finite (see [9]). If a group G is finitely presented as a group, then it is known that G is also finitely presented as a semigroup (see, [9]). In this paper we consider groups as semigroups. For a group G and its a normal subgroup N, the relation

$$\rho_N = \{(s,t) : st^{-1} \in N\}$$

defined on G is a congruence. Conversely, if ρ is a congruence on G, then the subset

$$N_\rho = \{st^{-1} : (s,t) \in \rho\}$$

of G is a normal subgroup of G (see, [6]). We note that given a normal subgroup N of a group G, since the congruence ρ_N on G is defined by $a\rho b$ if and only if $Na = Nb$, it follows that $G/N = G/\rho$. In this paper we prove that given a normal subgroup N of G with finite index, if G is finitely presented, the congruence ρ_N is finitely presented.

In the sequel, unless otherwise is stated, given a congruence relation ρ on a semigroup S by a generating set of ρ we mean a subset X of ρ which generates ρ as a subsemigroup in $S \times S$. We will explicitly state that ρ is generated by X as congruence if we mean that ρ is the smallest congruence in S containing X.

2. Presentation for ρ_N

We start with defining semigroup presentation. Let A be an alphabet, let A^+ be the free semigroup on A (i.e., the set of all non-empty words over A) and let A^* be the free monoid on A (i.e., A^+ together with the empty word, denoted by ε). A *semigroup presentation* is a pair $\langle A \mid R\rangle$ with $R \subseteq A^+ \times A^+$. A semigroup S is *defined by the presentation* $\langle A \mid R\rangle$ if S is isomorphic to the semigroup A^+/σ, where σ is the congruence on A^+ generated by R (i.e., the smallest congruence on A^+ containing R). For any two words $w_1, w_2 \in A^+$ we write $w_1 \equiv w_2$ if they are identical words and write $w_1 = w_2$ if $w_1\sigma = w_2\sigma$ (i.e., if they represent the same element in S). Therefore, the relation $w_1 = w_2$ holds in S if and only if this relation is a consequence of R, that is, there is a finite sequence $w_1 \equiv \alpha_1 \to \alpha_2 \to \cdots \to \alpha_k \equiv w_2$ of words from A^+, in which every term α_i $(1 < i \leq k)$ is obtained from α_{i-1} by applying one relation from R (see [5, Proposition 1.5.9]). A semigroup S is called *finitely presented* if S has a presentation $\langle A \mid R\rangle$ such that both A and R are finite.

We first find a generating set for the congruence ρ_N as a subsemigroup from a given generating set for N. Second we construct a presentation for ρ_N from a given presentation for N. Finally we conclude that ρ_N is finitely presented when G is finitely presented and the normal subgroup N has finite index in G.

Lemma 2.1. *Let N be a normal subgroup of a group G with index n. If G is finitely generated, then the congruence ρ_N is a finitely generated semigroup.*

Proof. Since N is a normal subgroup of index n, then there exist $u_1, \ldots, u_n \in G$ such that $G/N = \{Nu_1, Nu_2, \ldots, Nu_n\}$. Take $U = \{u_1, \ldots, u_n\}$ as a representative set of G/N. Suppose that a subset X of N is a generating set of N and that e is the identity element of G. Then we claim that the set

$$Y = \{(e,x), (x,e), (u,u) : x \in X, u \in U\}$$

is a generating set for ρ_N. To prove this claim we need to show that any element $(s,t) \in \rho_N$ can be written as a product of some elements of Y. Since $st^{-1} \in N$, there exists an element $u \in U$ such that $s, t \in Nu$ (where $Nu = Nt$). Since X is generating set for N, there exist $x_1, \ldots, x_k, y_1, \ldots, y_l \in X$ such that

$$s = x_1 \cdots x_k u \text{ and } t = y_1 \cdots y_l u.$$

Hence we have

$$(s,t) = (s,e)(e,t) = (x_1,e) \cdots (x_k,e)(e,y_1) \cdots (e,y_l)(u,u).$$

Since G is finitely generated and the N has finite index, N is finitely generated, and so there exists a finite generating set X for N. Since $|Y| = 2|X| + n$ is finite, ρ_N is finitely generated. □

Let $X = \{x_i : i \in I\}$ be a generating set for N, and let $U = \{u_1, \ldots, u_n\}$ be a representative set of G/N. If e is the identity element of G, then, for each $x_i \in X$, we denote the elements (x_i, e) and (e, x_i) by x_{1i} and x_{2i}, respectively, and denote the elements (u_j, u_j) of Y by v_j for each $1 \le j \le n$. Then we have just proved that $Y = X_1 \cup X_2 \cup X_3$ is a generating set for ρ_N where

$$X_1 = \{x_{11}, \ldots, x_{1m}\}, \quad X_2 = \{x_{21}, \ldots, x_{2m}\} \text{ and } X_3 = \{v_1, \ldots, v_n\}.$$

For a word $w \equiv x_{i_1} x_{i_2} \cdots x_{i_k} \in X^+$ we denote the words

$$x_{1i_1} \cdots x_{1i_k} \equiv (x_{i_1}, e) \cdots (x_{i_k}, e) \text{ and } x_{2i_1} \cdots x_{2i_k} \equiv (e, x_{i_1}) \cdots (e, x_{i_k})$$

by $\overline{w}$ and $\overline{\overline{w}}$, respectively.

Since N is normal, for any $u_i, u_j \in U$, there exists $u_{ij} \in U$ such that $(Nu_i)(Nu_j) = Nu_{ij}$. Thus we have a word $w_{ij} \in X^+$, which represents an element of N, such that the relation $u_i u_j = w_{ij} u_{ij}$ holds. Since $u_j x_i \in u_j N = Nu_j$ for any $u_j \in U$ and $x_i \in X$, there exists a word $w_{u_j, x_i} \in X^+$, which represents an element of N, such that the relation $u_j x_i = w_{u_j, x_i} u_j$ holds. We fix all w_{ij} and w_{u_j, x_i} which are given above. Now we state and prove the main theorem of this paper:

Theorem 2.2. *Let N be a normal subgroup of a group G with index n. With above notations if $\mathcal{P} = \langle X \mid R \rangle$ is a semigroup presentation for N, then $\mathcal{Q} = \langle Y \mid Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \rangle$ is a semigroup presentation of ρ_N where*

$$Q_1 = \left\{ \overline{r} = \overline{s},\ \overline{\overline{r}} = \overline{\overline{s}} \ : \ (r = s) \in R \right\},$$

$$Q_2 = \{ x_{2i} x_{1j} = x_{1j} x_{2i} \ : \ x_i, x_j \in X \},$$

$$Q_3 = \left\{ v_i v_j = \overline{w}_{ij} v_{ij},\ v_i v_j = \overline{\overline{w}}_{ij} v_{ij}\ :\ 1 \le i, j \le n \right\},$$
$$Q_4 = \left\{ v_i x_{1j} = \overline{w}_{u_i, x_j} v_i,\ v_i x_{2j} = \overline{\overline{w}}_{u_i, x_j} v_i\ :\ u_i \in U, x_j \in X \right\}.$$

Proof. From Lemma 2.1 we know that $Y = \langle X_1 \cup X_2 \cup X_3 \rangle$ is a generating set for ρ_N. It is routine to check that all the relations in $Q_1 \cup Q_2$ hold in ρ_N. And we have already explained that the relations in $Q_3 \cup Q_4$ hold in ρ_N. Therefore, ρ_N is a homomorphic image of the semigroup defined by the presentation $\mathcal{Q} = \langle Y \mid Q \rangle$ where $Q = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$.

For any word $w \in Y^+$, first of all, there exist words $s \in X_1^*$, $t \in X_2^*$ and $v \in X_3$ such that the relation $w = stv$ is a consequence of the relations from Q_4, Q_3 and Q_2, respectively. Let w_1 and w_2 be two words on Y representing the same element of ρ_N. Then there exist words $s_1, s_2 \in X_1^*$, $t_1, t_2 \in X_2^*$ and $v_1, v_2 \in X_3$ such that the relations

$$w_1 = s_1 t_1 v_1 \quad \text{and} \quad w_2 = s_1 t_2 v_2$$

are consequence of the relations in $Q_2 \cup Q_3 \cup Q_4$. Since the relation $w_1 = w_2$ hold in ρ_N, we must have the relations $s_1 = s_2$ and $t_1 = t_2$ holds in ρ_N, and the words v_1 and v_2 are identical, that is $v_1 \equiv v_2$. Thus, since the relations $s_1 = s_2$ and $t_1 = t_2$ are consequences of the relations in Q_1, it follows that the relation $s = t$ is consequence of Q. Therefore, $\mathcal{Q} = \langle Y \mid Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \rangle$ is a semigroup presentation of the congruence ρ_N on G. □

Corollary 2.3. *Let N be a normal subgroup of a group G with index n. If G is a finitely presented group, then the congruence ρ_N on the semigroup G is a finitely presented semigroup.*

Proof. Since N is a normal subgroup of a finitely presented group G with finite index, N is a finitely presented group, and so N is a finitely presented semigroup. Therefore, there exists a finite semigroup presentation $\mathcal{P} = \langle X \mid R \rangle$ for N. It follows from Lemma 2.1 and Theorem 2.2 that

$$\mathcal{Q} = \langle Y \mid Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \rangle$$

is a finite semigroup presentation for ρ_N, and so ρ_N is finitely presented. □

References

[1] I. M. Araujo, M. J. J. Branco, V. H. Fernandes, G. M. S. Gomes, and N. Ruskuc, *On generators and relations for unions of semigroups*, Semigroup Forum **63** (2001), no. 1, 49–62.

[2] H. Ayık and N. Ruskuc, *Generators and relations of Rees matrix semigroups*, Proc. Edinburgh Math. Soc. (2) **42** (1999), no. 3, 482–495.

[3] G. Ayık, H. Ayık, and Y. Ünlü, *Presentations for S and S/ρ from a given presentation ρ*, Semigroup Forum **70** (2005), no. 1, 146–149.

[4] C. Carvalho, R. D. Gray, and N. Ruskuc, *Presentations of inverse semigroups, their kernels and extension*, J. Aust. Math. Soc. **90** (2011), no. 3, 289–316.

[5] J. M. Howie, *Fundamentals of Semigroup Theory*, Clarendon Press, Oxford, 1995.

[6] J. M. Howie and N. Ruskuc, *Constructions and presentations for monoids*, Comm. Algebra **22** (1994), no. 15, 6209–6224.

[7] D. L. Johnson, *Presentations of Groups*, Cambridge University Press, Cambridge, 1990.
[8] T. G. Lavers, *Presentations of general products of monoids*, J. Algebra **204** (1998), no. 2, 733–741.
[9] E. F. Robertson, N. Ruskuc, and J. Wiegold, *Generators and relations of direct product of semigroups*, Trans. Amer. Math. Soc. **350** (1998), no. 7, 2665–2685.

Gonca Ayik
Department of Mathematics
Çukurova University
01330 Adana-Turkey
E-mail address: agonca@cu.edu.tr

Basri Çalışkan
Department of Mathematics
Osmanìye Korkut Ata University
Osmaniye-Turkey
E-mail address: bcaliskan@osmaniye.edu.tr