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Discrete ordinates (S_N) method for the first solution of the transport equation using Chebyshev polynomials

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Abstract. First estimates for the numerical solution of the one-dimensional neutron transport equation for one-speed neutrons in a finite homogeneous slab is studied. The neutrons are assumed to be scattered isotropically through the medium. Then the discrete ordinates form of the transport equation is solved for the eigenvalue spectrum using the Chebyshev polynomials of second kind in the neutron angular flux. Therefore, the calculated eigenvalues for various values of the c_0, the mean number of secondary neutrons per collision, are given in the tables using the Gauss-Chebyshev quadrature set.

1 Introduction

In the studies about neutron transport through any medium, the scattering of the neutrons should be taken into considerations carefully. Since the neutrons starts the fission in the system, it is important not to lose them to continue the fission chain reaction. Then, when the fission chain reaction begins to increase, it should be taken into a control to fix the power of the reactor. In other words, all reactors are wanted to produce constant energy, i.e. the multiplication factor is desired to be equal to one. This constant power situation can be sensed as isotropic scattering of the neutrons in the system. Therefore, it is still worth to study the isotropic scattering of the neutrons in transport theory.

There are many methods in literature to solve the discrete ordinates (S_N) transport equation numerically. Among them, the Monte Carlo (MCNP) and the source iteration (SI) techniques are extensively used in the solution algorithm of the transport equation. While the MCNP method is one of the first methods, the SI method is reported to converge rapidly for optically thin and slowly for optically thick problems [1,2]. Therefore, alternative methods are always looking for the numerical solution of the S_N transport equation.

In this study, the numerical solution of the S_N transport equation is investigated by using the second kind of Chebyshev quadrature sets. In the solution algorithm of the problem, first an analytic particular solution is derived by applying the separation of variables and then transport equation is converted in a discrete ordinates form which can be solved using Chebyshev quadrature.

2 Method

The stationary transport equation with isotropic scattering and constant neutron source can be written for one-speed neutrons [3],

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \sigma_T \psi(x,\ \ ) = \frac{\sigma_{S0}}{2} \int_{-1}^{1} \psi(x,\mu')\,\mathrm{d}\mu' + \frac{Q_0}{2} \tag{1}$$

where $\psi(x,\mu)$ is the angular flux of the neutrons at position x travelling in direction μ, cosine of the angle between the neutron velocity vector and the positive x-axis. σ_T is the total macroscopic cross-section, σ_{S0} is the differential scattering cross-section corresponding to isotropic scattering and Q_0 is the internal source. In order to solve Eq. (1), the integral term in this equation can be expressed as the integral transform with Chebyshev quadrature;

$$\int_{-1}^{1} \psi(x,\mu')\,\mathrm{d}\mu' = \int_{-1}^{1} \sqrt{1-\mu'^2}\,\psi(x,\mu') \frac{\mathrm{d}\mu'}{\sqrt{1-\mu'^2}} \cong \sum_{n=1}^{N} \frac{\psi_n(x)\omega_n}{\sqrt{1-\mu_n^2}} \tag{2}$$

Using Eq. (2) in Eq. (1), one can rearrange the discrete ordinates S_N equations for the numerical solution,

$$\mu_m \frac{\mathrm{d}\psi_m(x)}{\mathrm{d}x} + \sigma_T \psi_m(x) = \frac{\sigma_{S0}}{2} \sum_{n=1}^{N} \frac{\psi_n(x)\omega_n}{\sqrt{1-\mu_n^2}} + \frac{Q_0(x)}{2} \tag{3}$$

where ω_n is the Gauss-Chebyshev quadrature weights or weighting factor for direction μ_n, i.e. the roots of the Nth order Chebyshev polynomials of second kind. Therefore, the roots and the weighting factors of the Chebyshev polynomials of second kind can be given as respectively [4];

$$\mu_m = \cos\left(\frac{m\pi}{N+1}\right) \tag{4}$$

$$\omega_m = \frac{\pi}{N+1} \sin^2\left(\frac{m\pi}{N+1}\right) \quad m = 1,\ldots,N. \tag{5}$$

The general solution of Eq. (3) can be referred to as,

$$\psi_m(x) = \psi_m^p(x) + \psi_m^h(x) \tag{6}$$

where $\psi_m^p(x)$ and $\psi_m^h(x)$ denote the particular and the homogeneous solutions of Eq. (3), respectively. A spatially constant particular solution can easily be got from Eq. (3) with,

$$\alpha_N = \sum_{n=1}^{N} \frac{\omega_n}{\sqrt{1-\mu_n^2}} \tag{7}$$

$$\psi_m^p(x) = \frac{Q_0}{\sigma_T(1-c_0\alpha_N)}, \quad 0 \le a \le x, \qquad 1 \le m \le N \tag{8}$$

where $c_0 = \sigma_{S0}/\sigma_T$. The homogeneous solution $\psi_m^h(x)$ of Eq. (3) can be determined using the method of separation of variables. Therefore, the homogeneous solution can be written in the form of [5],

$$\psi_m^h(x) = H_m(\nu)\exp(\sigma_T x/\nu), \quad 0 \le a \le x, \qquad 1 \le m \le N \tag{9}$$

One can obtain an expression for the angular part of the neutron angular flux by using Eq. (9) in Eq. (3),

$$H_m(\nu) = \frac{\nu c_0}{2(\nu+\mu_m)} \sum_{n=1}^{N} \frac{H_n(\nu)\omega_n}{\sqrt{1-\mu_n^2}} \tag{10}$$

and the function $H_m(\nu)$ is normalized by,

$$\sum_{n=1}^{N} \frac{H_n(\nu)\omega_n}{\sqrt{1-\mu_n^2}} = 1 \tag{11}$$

In most of the studies in transport theory, the first step for those problems is to find the eigenvalue spectrum. For this purpose, Eq. (10) is multiplied by $\left(\omega_m / \sqrt{1-\mu_m^2}\right)$ from both sides and then summed over all m:

$$\sum_{m=1}^{N} \frac{\nu c_0}{2(\nu+\mu_m)\sqrt{1-\mu_m^2}} \omega_m = 1, \ \nu \neq -\mu_m \quad (12)$$

Eq. (12) is referred to as the dispersion relation and the roots $\nu_k, 1 \le k \le N$, of Eq. (12) are the eigenvalues of the S_N equations and they are lying symmetrically about the origin for any c_0 satisfying $0 \le c_0 \le 1$.

3 Numerical results and discussion

The one-speed and one-dimensional transport equation is investigated for the eigenvalue spectrum using S_N method with Gauss-Chebyshev (II. kind) quadrature. For this purpose, first the transport equation in the form of the integro-differential is reduced to S_N form by applying the integral transform with the even-order Gauss-Chebyshev quadrature set. Then, by solving the homogeneous part of the related equation, an analytic expression for the eigenvalues is obtained in Eq. (12). The roots $\nu_k, 1 \le k \le N$ of Eq. (12) are the eigenvalues of the S_N equations, i.e. Eq. (3).

In Table 1, calculated eigenvalues are given for various values of the $c_0 < 1$ and $c_0 > 1$. These eigenvalues are so computed to use different studies related with transport theory.

Table 1. Eigenvalue spectrum

N	$c_0 = 0.30$	$c_0 = 0.60$	$c_0 = 0.99$	$c_0 = 1.20$	$c_0 = 2.00$
2	±0.586037175	±0.740549631	±1.564262966	±1.682828200i*	±0.554257184i
4	±0.340237412	±0.379634162	±0.442276619	±0.476909013	±0.565264944
	±0.872064483	±1.016288788	±2.732719404	±1.309461630i	±0.457687467i
6	±0.225368072	±0.228117069	±0.231529477	±0.233286929	±0.239447540
	±0.650558027	±0.667286891	±0.679729369	±0.684024964	±0.693422727
	±1.125526328	±2.137618820	±1.245771884i	±0.933040613i	±0.554787056i

*$i = \sqrt{-1}$

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