Calculation of eigenvalues for neutron transport equation using Henyey-Greenstein phase function in slab geometry

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Abstract. Eigenvalues are obtained for one-dimensional steady-state neutron transport equation in slab geometry using Henyey-Greenstein (HG) phase function. Firstly, HG phase function is inserted into neutron transport equation then eigenvalues are calculated for different values of collision parameters *c* and *t* parameters. All results are calculated for P_9 and U_9 approximation and these results compared each other.

1 Introduction

In neutron transport problems, all scientist suggested valuable approximations and they solved many problems using various methods in different geometries. In the studies such as eigenvalue spectrum, criticality, flux problem, diffusion length, etc. they have also restrict to problem in the case of isotropic, anisotropic and strongly anisotropic. As well known, eigenvalues depend on the c values and the important coefficients such as diffusion length, diffusion coefficient and buckling also depend on the parameter *c*. In the system, as neutrons move complicated paths and these neutrons repeated nuclear collisions to describe an appropriate scattering function (phase function) is quite important in the solution of algorithm [1]. Scattering function gives the knowledge about behaviour of neutrons which can be absorbed or scattered in the reactor and as well known, as the incident neutron energy increases or the mass of interacting nucleus increases, neutron scattering becomes increasingly anisotropic [2]. In such cases, it is also difficult to predict the distribution of the neutrons in the system accurately. Up to now, different phase functions are used for the solution of the neutron transport equation and convenient results are obtained for each problem.

Henyey-Greenstein phase function (HG) is used in several studies to describe stellar light propagation throughout an atmosphere [3-4] and light scattering in the sea-water [5-6]. HG phase function is also used in bio-medical applications by some researchers [7-8]. Recently, HG phase function applied to neutron transport equation in slab geometry by some researchers and they showed that it is convenient to calculate critical half thicknesses [9]. In one of the latest study, Bülbül et al. calculated critical radius using HG phase function in spherical geometry with P_N approximation [10].

In this study, the eigenvalues have been calculated with U_N approximation using HG phase function in slab geometry. The eigenvalues are calculated for the 9th order of U_N approximation. Obtained numerical results are given in the tables and compared P_N approximation. One can see the effect of the scattering parameter *t* on the solution from the tables and can also decide the mathematical derivations are convenient for the neutron transport equation solutions.

2 Theory

In slab geometry, the steady-state time-independent neutron transport equation without sources is given as

$$\mu \frac{\partial \psi(x,\mu)}{dx} + \sigma_T \psi(x,\mu) = \frac{\sigma_S}{2} \int_{-1}^{1} \psi(x,\mu')\, d\mu', \qquad (1)$$
$$-a \le x \le a, \quad -1 \le \mu \le 1.$$

where $\psi(x,\mu)$ is the angular flux or flux density of neutrons at position *x* traveling in direction μ, σ_T and σ_S are total and scattering differential cross section, respectively. It is aimed to solve Equation (1) with HG phase function in this study and to do this, scientist use σ_S in terms of HG phase function and it is given as

$$\sigma_S{}^{HG}(\mu_0) = \frac{\sigma_S(1-t^2)}{4\pi\left(1-2\mu_0 t + t^2\right)^{3/2}} \qquad (2)$$

where σ_S is any non-negative coefficient, the parameter *t* is in the range of $0 \le t \le 1$ and $\mu_0 = \mathbf{\Omega}\cdot\mathbf{\Omega}'$ is the cosine of the scattering angel,

$$\mu_0 = \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\varphi-\varphi').$$

The steady state transport equation for one-dimensional case can be written when the HG phase function given in Eq. (2) is inserted on the right hand side of Eq. (1),

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$$\mu\frac{\partial\psi(x,\mu)}{\partial x}+\sigma_T\,\psi(x,\mu)= \int_{-1}^{1}\psi(x,\mu')\mathrm{d}\mu'\int_{0}^{2\pi}\frac{\sigma_s}{4\pi}\frac{(1-t^2)}{\left(1-2\mu_0 t+t^2\right)^{3/2}}\mathrm{d}\varphi' \tag{3}$$

Using

$$\int_{0}^{2\pi}\sigma_S^{HG}(\mu_0)\,\mathrm{d}\varphi'=\frac{\sigma_S}{2}\sum_{n=0}^{\infty}(2n+1)t^n P_n(\mu)P_n(\mu')$$

the equation can be written

$$\mu\frac{\partial\psi(x,\mu)}{\partial x}+\nu\,\psi(x,\mu)= \frac{\nu c}{2}\sum_{n=0}^{\infty}(2n+1)t^n P_n(\mu)\int_{-1}^{1}P_n(\mu')\psi(x,\mu')\mathrm{d}\mu' \tag{4}$$

To simplify the derivation of the equations, here a dimensionless space variable such that $\sigma_T x/\nu \rightarrow x$ is defined and ν is the eigenvalues. In order to solve Eq. (4), it is well known that in the U_N approximation the angular flux is expanded in terms of Chebyshev polynomial as

$$\psi(x,\mu)=\frac{2}{\pi}\sqrt{1-\mu^2}\sum_{n=0}^{N}\Phi_n(x)U_n(\mu), \quad -a\le x\le a,\ -1\le\mu\le 1 \tag{5}$$

If the neutron angular flux $\psi(x,\mu)$ given in Eq. (5) is inserted into Eq. (4), and the resulting equation is multiplied by $U_n(\mu)$ and integrated over $\mu \in [-1,1]$ using the orthogonality properties and the recurrence relations of Chebyshev polynomials given below

$$\int_{-1}^{1}U_n(\mu)U_m(\mu)\sqrt{1-\mu^2}\,d\mu=\begin{cases}\pi/2, & n=m\\ 0, & n\neq m\end{cases} \tag{6}$$

$$2\mu U_n(\mu)=U_{n+1}(\mu)+U_{n-1}(\mu),\quad -1\le\mu\le 1 \tag{7}$$

One can obtain the U_N moments of the angular flux for $n = 0$ and $n = 1$ respectively

$$\frac{\mathrm{d}\Phi_1(x)}{\mathrm{d}x}+2\nu\,\Phi_0(x)=2\nu c\Phi_0(x) \tag{8a}$$

$$\frac{\mathrm{d}\Phi_0(x)}{\mathrm{d}x}+\frac{\mathrm{d}\Phi_2(x)}{\mathrm{d}x}+2\nu\,\Phi_1(x)=2\nu ct\Phi_1(x) \tag{8b}$$

In order to obtain the eigenvalue spectrum, a well-known solution is employed of the form [11],

$$\Phi_n(x)=A_n(\nu,t)\,e^{\sigma_t x/\nu} \tag{9}$$

As well known in U_N approximation, the discrete and continuum ν eigenvalues can be obtained by setting $A_{N+1}(\nu,t) = 0$ for various values of c and t. For instance, one can obtain the eigenvalues for U_1 approximation letting $A_2(\nu,t) = 0$,

$$\nu_k=\pm\frac{1}{2\sqrt{(1-c)(1-ct)}} \tag{10}$$

From obtained eigenvalues, ν_k, $k = 1,\ldots,N + 1$, the general solution of the flux moments for odd numbers of N, i.e.

$$\Phi_n(x)=\sum_{k=1}^{\frac{N+1}{2}}\alpha_k A_n(\nu_k,t)\left[\exp\left(\frac{\sigma_T x}{\nu_k}\right)+(-1)^n\exp\left(-\frac{\sigma_T x}{\nu_k}\right)\right] \tag{11}$$

where the coefficients α_k can be determined from the physical boundary conditions of the system, and the parity relation of $A_n(-\nu,t) = (-1)^n A_n(\nu,t)$ is used. The other way to compute ν eigenvalues is to use the determinant of the coefficients matrix of $A_n(\nu,t) = 0$, that is

$$[\mathbf{M}(\nu)]\mathbf{A}(\nu,t)=0 \tag{12}$$

where $\mathbf{A}(\nu,t)$ is the column vector given by $[A_0(\nu,t), A_1(\nu,t), \ldots, A_{N+1}(\nu,t)]^T$ and $\mathbf{M}(\nu)$ is a $(N + 1)$ x $(N + 1)$ square matrix. If one solves the equation det $\mathbf{M}(\nu) = 0$ for any order $N+1$, it is clear that same results are obtained with the results of $A_{N+1}(\nu,t) = 0$.

Table 1. The eigenvalues for $c=1.01$ and $c=1.2$ and different values of t parameters

t	$c=1.01$		$c=1.2$	
	P_9	U_9	P_9	U_9
0.00	5.75053987I	5.75053987I	1.19826501I	1.19825343I
	0.944113891	0.916387696	0.947136093	0.920500197
	0.771095320	0.745729814	0.780103118	0.755077336
	0.503853642	0.478003149	0.515668192	0.488753622
	0.175087652	0.166360374	0.180883259	0.171664714
0.25	6.649411961	6.649411961	1.41957434I	1.41956964I
	0.948526547	0.922518128	0.954316077	0.930462699
	0.779202263	0.752627310	0.792491581	0.765337133
	0.510727741	0.484687660	0.525352617	0.498290747
	0.177697036	0.168655888	0.184263675	0.174579568
0.50	8.16232721I	8.16232721I	1.82098326I	1.82098176I
	1.001414356	0.986441109	1.040978655	1.031656343
	0.825302001	0.790568479	0.852946480	0.815217242
	0.542685181	0.514860964	0.567356918	0.537899614
	0.189058191	0.178697541	0.199060779	0.187562237
0.75	11.6159435I	11.6159435I	3.09223490I	3.09223487I
	1.421065877	1.418411784	1.921867006	1.921529396
	1.019227138	0.981682037	1.115927742	1.085095545
	0.680119610	0.640162449	0.755743458	0.708804312
	0.238384902	0.222912740	0.268552301	0.249212801
1.00	97.39065321	95.94929736	4.869532643	4.797464868
	86.50633604	84.12535328	4.325316833	4.206267664
	67.94095710	65.48607339	3.397047842	3.274303670
	43.33953939	41.54150130	2.166976971	2.077075065
	14.88743390	14.23148383	0.744371695	0.711574191

Table 2. The eigenvalues for *c=1.5* and *c=1.8* and different values of *t* parameters

t	*c=1.5*		*c=1.8*	
	P_9	U_9	P_9	U_9
0.00	0.68913625I	0.68901526I	0.50289477I	0.50258059I
	0.949871825	0.924354710	0.951455771	0.926634470
	0.789325683	0.764749794	0.795051896	0.770787789
	0.530823974	0.502907528	0.542085776	0.513738785
	0.190431649	0.180422216	0.200228911	0.189458321
0.25	0.84229018I	0.84224522I	0.63006128I	0.62997154I
	0.963703864	0.943353644	0.975481641	0.959025451
	0.811700416	0.783146576	0.830349978	0.799692774
	0.547487352	0.519336062	0.568444829	0.539728381
	0.195446003	0.184606249	0.207622727	0.195445564
0.50	1.16637397I	1.16637330I	0.96168368I	0.96178107I
	1.168140795	1.165643510	1.604543375	1.604406796
	0.894552174	0.854498094	0.930864256	0.893691006
	0.609460179	0.577613102	0.655157243	0.621058822
	0.217087207	0.203336724	0.238485840	0.221725882
0.75	------	------	------	------
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	1.398498024	1.386173187	1.119480408	1.064924851
	0.908153798	0.849845608	4.32752747I	4.32756475I
	0.335331613	0.305898020	0.446009951	0.395268803
1.00	1.947813057	1.918985947	1.217383157	1.199366218
	1.730126733	1.682507066	1.081329210	1.051566914
	1.358819137	1.309721468	0.849261962	0.818575918
	0.866790788	0.830830026	0.541744242	0.519268766
	0.297748678	0.284629677	0.186092924	0.177893548

3 Results and Discussion

The eigenvalue problem is studied using U_N approximation for the steady-state neutron transport equation without sources. To solve problem HG phase function is inserted into transport equation and eigenvalues are obtained for U_9 and P_9 approximation. All computations are carried out using Maple software and the total macroscopic cross section is assumed to be its normalized value, $\sigma_T = 1$ cm^{-1}. The eigenvalues can be calculated by setting $A_{N+1}(\nu,t) = 0$ for various values of *c* and *t*. Numerically calculated eigenvalues for $c > 1$ are given in the Table and one pair of the roots is observed purely imaginary the others are real for increasing *t* parameters from 0.00 to 0.75. The whole eigenvalues for *t=1* are real and for *c=1.5, c=1.8*, *t=0.75*, physically all eigenvalues are existing but mathematically it could not be calculated. We can say that HG phase function works for the calculation of the eigenvalues in slab geometry for neutron transport equation. The aim is in this study that to show applicability of HG phase function to neutron transport equation in slab geometry using U_N approximation and to calculate eigenvalues which can be used for the calculation of criticality, scalar flux, albedo problem, etc.

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